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An orbifold relative index theorem

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Abstract

In this paper we prove a relative index theorem for pairs of generalized Dirac operators on orbifolds which are the same at infinity. This generalizes to orbifolds a celebrated theorem of Gromov and Lawson.

MSC: primary 58G03; secondary 58G10

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0. Introduction

Orbifolds, generalized manifolds that are locally the quotient of a euclidean space modulo a finite group of isometries, were first introduced by Satake. In the late seventies, Kawasaki proved an orbifold signature formula, together with more general index formulas; see [20–22]. In [10] we proved a K-theoretical index theorem for orbifolds with operator algebraic means, and in [11,12] we studied compact orbifold spectral theory and defined orbifold eta invariants. Other orbifold index formulas were proved in [9,31]. In [8] Chiang studied compact orbifold heat kernels and harmonic maps, while in [29], Stanhope established some interesting geometrical applications of orbifold spectral theory.

In [13] we continued our orbifold spectral analysis started with [11,12]. In particular we showed that on a non-compact complete almost complex Spin^c orbifold which is sufficiently regular at infinity (see Definition 2.1), generalized Dirac operators are closed. This generalizes to orbifolds theorems of Gaffney [16], Yau [30], and Wolf [32] for the manifold case. We also proved an orbifold divergence/Stokes theorem.

Here we will use the results we proved in [13] to establish an orbifold Gromov–Lawson, [15], relative index theorem for non-compact complete almost complex Spin^c orbifolds which are sufficiently good at infinity (see the definition right before the statement of Theorem 3.5). Further generalizations to orbifolds of theorems in the spirit of the main result in [2,3,18] will be considered in a later paper, [14]. We will also plan on investigating the question of whether every open complete orbifold is sufficiently good at infinity, which we suspect has a negative answer.

The relative index theorem applies to pairs of + generalized Dirac operators which agree at infinity. The topological index of such a pair is defined to be the difference in index of two natural extensions of the given operators to

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closed manifolds; see Definition 4.1. Kawasaki's theorems for closed orbifolds make it possible to express the pair topological index via local traces and orbifold characteristic classes. The topological index is calculated by using pairs of parametrices that agree at infinity; cf. Theorem 4.4.

The analytical index of a $+$ generalized Dirac operator, proved to be finite in Section 3, is defined, classically, to be the dimension of the kernel minus the dimension of the cokernel. The analytical index of a pair is defined to be the difference of the analytical indices of the pair's constituents; see Definition 3.6.

Our main result is Theorem 6.2, which asserts the equality of the analytical and topological indices of a pair of generalized Dirac operators that agree at infinity, which generalizes to orbifolds the main theorem of [15].

Theorem 6.2. *Let X be an even dimensional non-compact complete almost complex Hermitian* Spin^c *orbifold which is sufficiently good at infinity. Assume that a Hermitian connection is chosen on the dual of its canonical line bundle. Let D_i and $D_i^{\pm}$ be the generalized Dirac operators on X with coefficients in the proper Hermitian orbibundle E_i (with connection ∇^{E_i}), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood Ω of infinity. Assume that there exists a constant $k_0 > 0$ such that*

$$\mathcal{R} \geq k_0 \,\mathrm{Id} \quad \text{on } \Omega,$$

where $\mathcal{R}$ is given as in Proposition 2.7. Then the topological index (as in Definition 4.1) and the analytical index (as in Definition 3.6) of the pair (D_1^+, D_2^+) coincide, that is

$$\mathrm{ind}_t(D_1^+, D_2^+) = \mathrm{ind}_a(D_1^+, D_2^+).$$

Our proof is a generalization to orbifolds of the argument given for manifolds by Gromov and Lawson in [15]. New techniques are mainly introduced to deal with orbifold distance functions.

In more detail, the contents of this paper are as follows. In Section 1 we recall the definitions of orbifolds and orbibundles, and introduce orbifold generalized Dirac operators. In Section 2 we recall properties of generalized Dirac operators on non-compact orbifolds from [13]. In Section 3 we prove that generalized Dirac operators (on complete sufficiently regular orbifolds) which are positive at infinity have finite dimensional kernels and cokernels. By using these results we can thus define the analytical index of a pair of generalized Dirac operators that agree at infinity as the difference of their analytical indices; see Definition 3.6. In Section 4 we define the pair topological index by using a gluing technique and characterize it by using Kawasaki's index theorem for closed orbifolds; see Definition 4.1 and Theorem 4.2. In Theorem 4.4 we show how to compute the pair topological index by using parametrices that agree at infinity. Section 5 is devoted to detailing some local properties of traces of generalized Dirac operators. Finally, in Section 6 we state and prove our main result, the orbifold relative index theorem.

In the sequel, all orbifolds and manifolds are assumed to be even dimensional, smooth, Hermitian, Spin^c, connected, and almost complex unless otherwise specified. All vector and orbibundles are assumed to be smooth and proper. We also assume that all of our orbifolds/manifolds are endowed with a fixed connection on the dual of their canonical line bundle K^*. This allows us to define a 'canonical' Spin^c Dirac operator and, given a Hermitian orbibundle E endowed with a connection, the 'canonical' Spin^c Dirac operator with coefficients in E. Both of these operators depend, in the Spin^c case, on the choice of the selected connections; see [9, Chapter 14], and [23, Appendix D]. For the Spin or complex case, the choice of the connection on K^* is canonical.

1. Orbifolds, orbibundles and Dirac operators

In this section we will review some definitions and results that we will use throughout this paper. For generalities on orbifolds and operators on orbifolds; see [20–22,6,8,9].

An orbifold is a Hausdorff second countable topological space X together with an atlas of charts $\mathcal{U} = \{(\tilde{U}_i, G_i) | i \in I\}$, with $\tilde{U}_i / G_i = U_i$ open and with projection $\pi_i : \tilde{U}_i \to U_i$, $i \in I$, satisfying the following properties:

(1) If two charts U_1 and U_2 associated with pairs $(\tilde{U}_1, G_1)$, $(\tilde{U}_2, G_2)$ of $\mathcal{U}$, are such that $U_1 \subseteq U_2$, then there exists a smooth open embedding λ: $\tilde{U}_1 \to \tilde{U}_2$ and a homomorphism μ: $G_1 \to G_2$ such that $\pi_1 = \pi_2 \circ \lambda$ and $\lambda \circ \gamma = \mu(\gamma) \circ \lambda$, $\forall \gamma \in G_1$.
(2) The collection of the open charts U_i, $i \in I$, belonging to the atlas $\mathcal{U}$ forms a basis for the topology on X.

We will call an orbifold atlas as above a standard orbifold atlas.

For any x point of X, the isotropy G_x of x is well defined, up to conjugacy, by using any local coordinate chart. The set of all points $x \in X$ with non-trivial stabilizer, $\Sigma(X)$, is called the singular locus of X. $\Sigma(X)$ is a set of codimension at least 2; see e.g. [8]. Note that $X - \Sigma(X)$ is a manifold.

If we now endow X with a countable locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(\tilde{U}_i, G_i)|i \in \mathbb{N}\}$, then by standard theory there exists a smooth partition of unity $\eta = \{\eta_i\}_{i\in\mathbb{N}}$ subordinated to $\mathcal{F}$, [8]. This in particular means that, for any $i \in \mathbb{N}$, η_i is a smooth function on U_i (i.e., its lift to any chart of a standard orbifold atlas is smooth); the support of η_i is included in an open subset U_i' of U_i, and $\cup U_i' = X$. We will call any η as above an $\mathcal{F}$-partition of unity.

Let E be a Hermitian orbibundle (with connection ∇^E) over the orbifold X. (For the precise definition see [20–22, 8].) In particular E is an orbifold in its own right; on an orbifold chart U_1 associated with a pair $(\tilde{U}_1, G_1)$ of a standard orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i)|i \in I\}$ of X, E lifts to a G_1-equivariant bundle. Standard orbifold atlases on X can be used to provide standard orbifold atlases on E.

If E is an orbibundle over the orbifold X, a section $s : X \to E$ is called a smooth orbifold section if for each chart U_i associated with a pair $(\tilde{U}_i, G_i)$ of a standard orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i)|i \in I\}$ of X, we have that $s|_{U_i} : U_i \to E|_{U_i}$ is covered by a smooth G_i-invariant section $\tilde{s}|_{\tilde{U}_i} : \tilde{U}_i \to \tilde{E}|_{\tilde{U}_i}$. Given an orbibundle E over X, we will denote by $\mathcal{C}^\infty(X, E)$ the space of all smooth sections of E, and by $\mathcal{C}_c^\infty(X, E)$ the space of all smooth sections with compact support. Classical orbibundles over X are the tangent bundle TX, and the cotangent bundle T^*X of X. We can form orbibundle tensor products by taking the tensor products of their local expressions in the charts of a standard orbifold atlas.

Define an inner product between sections of $\mathcal{C}^\infty(X, E)$ (or $\mathcal{C}_c^\infty(X, E)$) by the following formula (cf., [8, 2.2a]):

$$(\sigma_1, \sigma_2) = \sum_{i=1}^{+\infty} \frac{1}{|G_i|} \int_{\tilde{U}_i} \tilde{\eta}_i(\tilde{x}_i)\langle\tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i)\rangle \mathrm{d}v(\tilde{x}_i),$$

where $\eta = \{\eta_i\}_{i\in\mathbb{N}}$ is an $\mathcal{F}$-partition of unity subordinated to the locally finite orbifold cover $\mathcal{F} = \{(\tilde{U}_i, G_i)|i \in \mathbb{N}\}$, and $\langle,\rangle$ is a G_i-invariant product on $\tilde{E}$. (Note that, with slight abuse of notation, we used $\tilde{}$ to denote lift to $\tilde{U}_i$.)

We will now review the construction of the generalized Dirac operator with coefficients in a Hermitian orbibundle E (with connection ∇^E) over a (compact or not) orbifold X; see [9, Sections 5 and 12], [21,5], and [23, Appendix D] in the manifold case. First of all, X admits a Spin^c-principal tangent orbibundle, Spin$^c(TX)$, with, under our hypotheses, a canonical Spin^c orbifold connection ∇^c. Let $\Delta^{\pm,c}$ be the half-Spin^c representations (recall that the X is even dimensional). Then we have two orbibundles

$$\Delta^{\pm,c}(TX) = \mathrm{Spin}^c(TX) \times_{\mathrm{Spin}^c} \Delta^{\pm,c},$$

with induced connections $\nabla^{\pm,c}$, from ∇^c; $\nabla^{\pm,c} : \mathcal{C}_c^\infty(X, \Delta^{\mp,c}(TX)) \to \mathcal{C}_c^\infty(X, T^*X \otimes \Delta^{\mp,c}(TX))$.

The Clifford module structure on $\Delta^{\pm,c}$ defines Clifford multiplications

$$m_\pm : TX \otimes_{\mathbb{R}} \Delta^{\pm,c}(TX) \to \Delta^{\mp,c}(TX).$$

Then the generalized $\pm$ Dirac operator with coefficients in E, $d_E^{\pm,c}$,

$$d_E^{\pm,c} : \mathcal{C}_c^\infty(X, \Delta^{\pm,c}(TX) \otimes_{\mathbb{C}} E) \to \mathcal{C}_c^\infty(X, \Delta^{\mp,c}(TX) \otimes_{\mathbb{C}} E),$$

is defined by

$$d_E^{\pm,c} = M \circ \left(\nabla^{\pm,c} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^E\right),$$

where M denotes the map induced by Clifford multiplication and TX has been identified with T^*X via the orbifold metric. We will also use the notation $\mathcal{S}$ for the orbifold Spin^c bundle $(\Delta^{+,c} \oplus \Delta^{-,c})(TX)$, and $\mathcal{S} \otimes E$ or $\mathcal{E}$ for $(\Delta^{+,c} \oplus \Delta^{-,c})(TX) \otimes_{\mathbb{C}} E$. We will define D_E, the generalized Dirac operator on X with coefficient in E, to be $(d_E^{+,c} + d_E^{-,c})$.

2. Dirac operators on non-compact complete orbifolds

On an orbifold X satisfying our hypotheses (but not necessarily compact), the generalized Dirac operator D_E with coefficients in the orbibundle E, as defined in the Section 1, is given by

$$D_E : \mathcal{C}_c^\infty(X, (\Delta^{+,c} \oplus \Delta^{-,c})(TX) \otimes_{\mathbb{C}} E) \to \mathcal{C}_c^\infty(X, (\Delta^{-,c} \oplus \Delta^{+,c})(TX) \otimes_{\mathbb{C}} E)$$
$$D_E = M \circ \left((\nabla^{+,c} + \nabla^{-,c}) \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^E \right).$$

On orbifold charts, D_E has the following local expression, $\tilde{D}_E$. Let $\mathcal{U} = \{(\tilde{U}_i, G_i) | i \in I\}$, with $\tilde{U}_i/G_i = U_i$, be a standard orbifold atlas. On a local chart $\tilde{U}_i$, $i \in I$ fixed, we have

$$\Delta^{\pm,c}(T\tilde{U}_i) = \mathrm{Spin}^c(T\tilde{U}_i) \times_{\mathrm{Spin}^c} \Delta^{\pm,c},$$

with induced G_i-invariant connections $\nabla^{\pm,c}$, from ∇^c. The Clifford module structure on $\Delta^{\pm,c}$ defines Clifford multiplications

$$m_\pm : T\tilde{U}_i \otimes_{\mathbb{R}} \Delta^{\pm,c}(T\tilde{U}_i) \to \Delta^{\mp,c}(T\tilde{U}_i).$$

On $\tilde{E}$, the lift of E, we have the G_i-invariant connection $\nabla^{\tilde{E}}$. Then the generalized $\pm$ Dirac operators with coefficients in E, $\tilde{d}_E^{\pm,c}$,

$$\tilde{d}_E^{\pm,c} : \mathcal{C}_c^\infty(\tilde{U}_i, \Delta^{\pm,c}(T\tilde{U}_i) \otimes_{\mathbb{C}} \tilde{E}) \to \mathcal{C}_c^\infty(\tilde{U}_i, \Delta^{\mp,c}(T\tilde{U}_i) \otimes_{\mathbb{C}} E),$$

are given by

$$\tilde{d}_E^{\pm,c} = M \circ \left(\nabla^{\pm,c} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\tilde{E}} \right),$$

where M is induced by Clifford multiplication and $T\tilde{U}_i$ has been identified with $T^*\tilde{U}_i$ via the G_i-invariant metric. Also, $\tilde{D}_E$, the generalized Dirac operator on X with coefficient in E, is given by $\tilde{d}_E^{+,c} + \tilde{d}_E^{-,c}$ on $\tilde{U}_i$.

If $e_1, \ldots, e_n$ is an orthonormal local basis for the space $T\tilde{U}_i$ at a point $\tilde{x}$, then $\tilde{D}_E$ has local expression

$$\tilde{D}^E = \sum_{k=1}^n e_k \tilde{\nabla}_{e_k}^E,$$

where

$$\tilde{\nabla}^E = (\nabla^{+,c} + \nabla^{-,c}) \otimes 1 + 1 \otimes \nabla^{\tilde{E}}.$$

By analogy with the manifold case, see [15,32,16,23,28,30], one can show that D_E is symmetric, whenever X is sufficiently regular at infinity.

Definition 2.1. Let X be a non-compact complete orbifold. Then we say that X is sufficiently regular at infinity if, for any neighborhood $\Omega \subseteq X$ of infinity, there exists a compact domain K_Ω with boundary strictly included in Ω, on which the divergence and Stokes theorems hold.

For a compact orbifold without boundary, the divergence theorem holds, [8]. See also [7] for other results. Sufficient regularity also holds in the case of a product end, by an adaptation of Chiang's method; see [8]. In general, ours seems to be a very reasonable assumption to make, which will certainly be satisfied in many cases of interest. For Sobolev inequalities of Gallot type involving domains, see [24].

Theorem 2.2 ([13, Theorem 2.2]). *Let X be a non-compact complete orbifold which is sufficiently regular at infinity, and let E be a Hermitian orbibundle over X. Let D_E be the generalized Dirac operator with coefficients in E, as defined above. Then D_E is symmetric, i.e.,*

$$(D_E\sigma_1, \sigma_2) = (\sigma_1, D_E\sigma_2), \quad \forall \sigma_1, \sigma_2 \in \mathcal{C}_c^\infty(X, \mathcal{S} \otimes_{\mathbb{C}} E),$$

where $(,)$ denotes the inner product defined in the introduction. (Actually, to obtain the above equality, it is enough to assume that only one of the sections σ_1, $i = 1, 2$, has compact support.)

Now complete the space $\mathcal{C}_c^\infty(X,\mathcal{E})$, $\mathcal{E} = \mathcal{S}\otimes_{\mathbb{C}} E, \mathcal{S}$ a Spin^c-bundle on X, E a Hermitian orbibundle (with connection ∇^E) over X, with respect to the norm

$$\|\sigma\|_X = \sqrt{\langle\sigma,\sigma\rangle} = \left(\sum_{i=1}^{+\infty}\frac{1}{|G_i|}\int_{\tilde{U}_i}\tilde{\eta}_i(\tilde{x}_i)\langle\tilde{\sigma}_1(\tilde{x}_i),\tilde{\sigma}_2(\tilde{x}_i)\rangle \mathrm{d}v(\tilde{x}_i)\right)^{\frac{1}{2}}.$$

We thus obtain the $\mathcal{L}^2$-space $\mathcal{L}^2(X,\mathcal{E})$. The generalized Dirac operator:

$$D_E : \mathcal{C}_c^\infty(X,\mathcal{E}) \to \mathcal{C}_c^\infty(X,\mathcal{E})$$

has two natural extensions, min and max, see [13], as an unbounded operator

$$D_E : \mathcal{L}^2(X,\mathcal{E}) \to \mathcal{L}^2(X,\mathcal{E}).$$

Theorem 2.3 (*[13, Theorem 3.1]*). *Let X be a non-compact complete orbifold which is sufficiently regular at infinity, let E be a Hermitian orbibundle (with connection ∇^E) over X, and let D_E be the generalized Dirac operator with coefficients in E. Let $\mathcal{D}(D_E^{\mathrm{MIN}})$ be the domain of the min extension of D_E, and $\mathcal{D}(D_E^{\mathrm{MAX}})$ be the domain of the max extension of D_E. Then*

$$\mathcal{D}(D_E^{\mathrm{MIN}}) = \mathcal{D}(D_E^{\mathrm{MAX}}).$$

The following very useful proposition was proved in [13]; it is a generalization of results proved by Gaffney and Yau for manifolds; see [17,30].

Proposition 2.4 (*[13, Proposition 3.4]*). *Let X be a non-compact complete orbifold which is sufficiently regular at infinity, and let $y_0 \in X - \Sigma(X)$ be a fixed point of X. Then there exists a sequence of continuous functions b_k, $k \in \mathbb{N}$, with:*

(1) $b_k : X \to [0, 1]$.
(2) $b_k = 1$ *on* $B_k = \{y \in X | \rho(y) = d(y, y_0) \le k\}$.
(3) *The support of b_k is contained in $\overline{B}_{2k}$.*
(4) *The function b_k is differentiable almost everywhere and at points of differentiability we have*

$$\|\nabla(b_k)\|^2 \le \frac{M^2}{k^2}, \quad k \in \mathbb{N}.$$

In [13] we also proved the following results.

Theorem 2.5 (*[13, Theorem 4.1]*). *Let X be non-compact complete orbifold which is sufficiently regular at infinity, and let E be a Hermitian orbibundle (with connection ∇^E) over X. Let*

$$D_E : \mathcal{C}_c^\infty(X,\mathcal{E}) \to \mathcal{C}_c^\infty(X,\mathcal{E}),$$

be the generalized Dirac operator on X with coefficients in E. Then $D_E(\sigma) = 0$ if and only if $D_E^2(\sigma) = 0$ for any $\sigma \in \mathcal{D}(D_E)$.

Proposition 2.6 (*[13, Proposition 6.1]*). *Let X be a non-compact complete* Spin^c *orbifold which is sufficiently regular at infinity, and let $\mathcal{S}$ be the* Spin^c *bundle of X. Then for any two sections σ_j, $j = 1, 2$, in $\mathcal{C}^\infty(X,\mathcal{S})$, at least one of which has compact support, we have*

$$\int_X \langle\Delta\sigma_1,\sigma_2\rangle \mathrm{d}v = \int_X \langle\nabla\sigma_1,\nabla\sigma_2\rangle \mathrm{d}v.$$

Proposition 2.7 (*[13, Proposition 6.3]*). *Let X be an orbifold. Let D be the Dirac operator on X, and let Δ be the* Spin^c *Laplacian. Then*

$$D^2 = \Delta + \mathcal{R}$$

where $\mathcal{R}$ is given below (cf. [9, *Theorem 6.1*], [23, *Theorem D12*] *for the manifold case),*

$$\mathcal{R} = \frac{1}{4}k + \frac{1}{2}c(K^*),$$

where k is the scalar curvature, and $c(K^*)$ denotes the curvature on the connection in the line bundle K^*.

When D_E is the generalized Dirac operator on X with coefficients in the proper Hermitian orbibundle (with connection ∇^E) E, then the formulas above become

$$D_E^2 = \Delta_E + \mathcal{R}_E,$$
$$\mathcal{R}_E = \frac{1}{4}k + \frac{1}{2}c(K^*) + c(E),$$

where $c(K^*)$ denotes Clifford multiplication of the curvature 2-form of the fixed connection on the line bundle K^*, and $C(E)$ is the Clifford multiplication of the curvature 2-form of the fixed connection ∇^E on E. When X is Spin, we can assume that $c(K^*) = c(E) = 0$, and $\mathcal{R}_E = \Delta_E + \frac{1}{4}k$.

Theorem 2.8 (*[13, Theorem 6.4]*). *Let X be a complete, non-compact,* Spin^c *orbifold which is sufficiently regular at infinity. If D is the Dirac operator on X with coefficients in the* Spin^c *bundle $\mathcal{S}$, then the domain $\mathcal{D}$ of the unique self-adjoint extension of D is exactly*

$$\mathcal{L}^{1,2}(X, \mathcal{S}), \quad \text{that is},$$

the completion of $\mathcal{C}_c^\infty(X, \mathcal{S})$ in the norm

$$\|\sigma\|_1^2 = \int_X (\langle\sigma, \sigma\rangle + \langle\nabla\sigma, \nabla\sigma\rangle)\,\mathrm{d}v = \int_X (\langle\sigma, \sigma\rangle + \langle\Delta_\sigma, \sigma\rangle)\,\mathrm{d}v.$$

Moreover, for every $\sigma \in \mathcal{D}$,

$$\|D\sigma\|_X^2 = \|\nabla\sigma\|_X^2 + (\mathcal{R}\sigma, \sigma),$$

where $\|\,\|_Y$ denotes the $\mathcal{L}^2(Y, \mathcal{E})$, $Y \subseteq X$, norm, $\mathcal{R}$ is given as in Proposition 2.7, and $(,)$ is the $\mathcal{L}^2$ inner product.

3. Dirac operators and Green's operators

As pointed out by Agmon in [1, Section 6], Friedrichs's lemma is a local result. Hence we can also assume it holds for orbifolds. (This follows from the local version of the manifold lemma applied to covers of orbifold charts of a locally finite orbifold cover.)

Theorem 3.1 (*Friedrichs's Lemma for Orbifolds*). *Let X be an orbifold. Let $\mathcal{S}$ be the* Spin^c *bundle of X, and let D be the Dirac operator on X, $D : \mathcal{C}_c^\infty(X, \mathcal{S}) \to \mathcal{C}_c^\infty(X, \mathcal{S})$. Let Ω be an open set in X and let K be a compact subset of Ω. Let $k \in \mathbb{N}$. Then there exists a constant C, depending only on K, Ω, and k, such that, for any $\sigma \in \mathcal{C}_c^\infty(X, \mathcal{S}|_\Omega)$ with $D\sigma = 0$, we have*

$$\|\sigma\|_{\mathcal{C}^k, K} \le C\|\sigma\|_{\mathcal{L}^2(\Omega, \mathcal{S})},$$

where $\|\,\|_{\mathcal{C}^k, K}$ is the uniform $\mathcal{C}^k$ norm for sections on K.

Proof. See [15, Theorem 3.7] for the manifold case. Endow X with a countable locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(\tilde{U}_i, G_i) | i \in \mathbb{N}\}$, with associated partition of unity $\eta = \{\eta_i\}_{i\in\mathbb{N}}$ subordinated to $\mathcal{F}$, [8]. Suppose that $U_i \cap K \neq \emptyset$ for only $i = 1, \ldots, \ell$. We can also choose our atlas so that $\cup_{i=1}^{\ell} U_i \subseteq \Omega$. Then by the manifold local version of Friedrichs's lemma [1, Section 6], modulo multiplication by $|G_i|$, $i = 1, \ldots, \ell$, we obtain the claim. □

We can now prove that Dirac operators which are positive at infinity have finite dimensional kernels and cokernels.

Theorem 3.2. *Let X be a non-compact complete orbifold which is sufficiently regular at infinity. Let $\mathcal{S}$ be the* Spin^c *bundle of X, and let D be the Dirac operator on X, $D : \mathcal{C}_c^\infty(X, \mathcal{S}) \to \mathcal{C}_c^\infty(X, \mathcal{S})$. Assume that there exists a compact subset K of X such that*

$$\mathcal{R} \ge k_0\,\mathrm{Id}$$

in $X - K$, where $\mathcal{R}$ is given as in Proposition 2.7. Then there exists an integer d depending only on D on a neighborhood of K and on k_0 such that

$$\dim(\mathrm{Ker}D) \le d.$$

In particular, if the dimension of X is even, and $D = D^+ \oplus D^-$ (see Section 1), then

$$\dim(\mathrm{Ker}D^+) + \dim(\mathrm{Ker}D^-) \le d.$$

Proof. Since K is compact, there exists a $k_1 > 0$ such that $k_1\mathrm{Id} \ge -\mathcal{R}$ in K. Now let $\sigma \in \mathcal{L}^2(X, \mathcal{S})$, with $D\sigma = 0$. Firstly, $\sigma \in \mathcal{C}^\infty(X, \mathcal{S})$, since solutions of elliptic equations on manifolds are smooth by a local argument. Then by Propositions 2.6 and 2.7, we have, for σ with compact support,

$$\|\nabla\sigma\|_X^2 + (\mathcal{R}\sigma, \sigma) = 0,$$

where $\| \|_Y$, $Y \subseteq X$, is the $\mathcal{L}^2(Y, \mathcal{S})$ norm. Since $\mathcal{R} \ge k_0$ on $X - K$, we have

$$\|\nabla\sigma\|_X^2 + \int_K \langle\mathcal{R}\sigma, \sigma\rangle + k_0 \int_{X-K} \|\sigma\|^2 \le 0.$$

Therefore,

$$\|\nabla\sigma\|_X^2 + k_0\|\sigma\|_{X-K}^2 \le \|\sigma\|_K^2.$$

We now make the assumption that $\|\sigma\|_X^2 = \|\sigma\|_K^2 + \|\sigma\|_{X-K}^2 = 1$. On adding $k_0\|\sigma\|_K^2$ to both sides, and dividing by $k_0 + k_1$, the above inequality becomes

$$\frac{\|\nabla\sigma\|_X^2}{k_0 + k_1} + \frac{k_0}{k_0 + k_1} \le \|\sigma\|_K^2. \tag{3.1}$$

Fix a neighborhood Ω of K and let C be the constant appearing in Theorem 3.1 for $k = 1$. Fix $\epsilon > 0$ and an ϵ dense subset $\{x_s\}_{s=1,\ldots,d}$ of K so that every point in K is within distance ϵ of some x_s. Let $H = \mathrm{Ker}(D)$ on $\mathcal{L}^2(X, \mathcal{S})$ and suppose by contradiction that $\dim(H) > d$. Then there exists $\sigma \in H$ such that $\|\sigma\|_X = 1$, and $\sigma(x_s) = 0$, $s = 1, \ldots, q$. By Theorem 3.1, applied to K and Ω, and the mean value theorem applied on lifts of orbifold charts, we have

$$\|\sigma(x)\| \le \sup_K \|\nabla(\sigma)\|\epsilon \le C\|\sigma\|_\Omega\epsilon.$$

But this contradicts (3.1). (Note that we can assume that σ has compact support since we are only interested in behavior near K.) □

By applying Glazman's variational lemma below, see [19, Proposition 3.4] or [27], we can obtain information on the spectrum of the Dirac operator.

Proposition 3.3. *Let A be a self-adjoint Hilbert operator that is semi-bounded from below. Let $N_h(\lambda)$ denote the number of eigenvalues in $(-\infty, \lambda]$, with multiplicity, and counting points of the continuous spectrum as points with infinite multiplicity. Then*

$$N_h(\lambda) = \sup_H \dim H,$$

where the supremum is taken over all the subspaces H which are such that

$$\langle Ah, h\rangle \le \langle h, h\rangle, \quad \forall h \in H.$$

Corollary 3.4. *Let X, D, and $\mathcal{S}$ be as in Theorem 3.2. Then D^2, and consequently D, have essential spectrum separated from* 0.

We will now show that Dirac operators admit Green's operators; cf. [15, Theorem 3.7], and [19, Propositions 3.3 and 3.4] for the manifold case. We assume here that the orbifold X is sufficiently good at infinity. This means that, for any neighborhood of infinity Ω, we can chop off X along an orbifold hypersurface, O_Ω, which is the boundary of a neighborhood of infinity included in Ω. We also assume that Ω is of product type near O_Ω.

Theorem 3.5. *Let X be a non-compact complete* Spin^c *orbifold which is sufficiently good at infinity. Let $\mathcal{S}$ be the* Spin^c *bundle of X, and let D be the Dirac operator on X, $D : \mathcal{C}_c^\infty(X, \mathcal{S}) \to \mathcal{C}_c^\infty(X, \mathcal{S})$. Assume that there exists a compact subset K such that*

$$\mathcal{R} \geq k_0 \mathrm{Id}$$

on $X - K$, where $\mathcal{R}$ is given as in Proposition 2.7. Assume that H is the finite dimensional kernel of D on $\mathcal{L}^2(X, \mathcal{S})$, and let $H^\perp$ be its orthogonal complement. Then there is a $\alpha > 0$ such that

$$\|D\sigma\|_X^2 \geq \alpha^2 \|\sigma\|_X^2, \quad \forall \sigma \in H^\perp.$$

Thus the operator D, and also, since X is even-dimensional, the operator $D^\pm$, admits bounded Green's operators.

Proof (*Cf. [15, Proof of Theorem 3.7] and [19, 3.3 and 3.4]*). First, we claim that D has only a point spectrum. This is proved by the same argument as was used in [19, 3.3]. Note that Kucerovsky's proof works verbatim in the orbifold case too, since Rellich's lemma applies in the sufficiently good case (in fact, one can double the domain to get an orbifold, and then apply Rellich's lemma for closed orbifolds proved in [11]). Now, let E_λ be the λ-eigenspace of D on $\mathcal{L}^2(X, \mathcal{S})$ with eigenvalue λ. It will suffice to prove that there exists an $\alpha > 0$ such that the space

$$H_\alpha = \oplus_{|\lambda| \leq \alpha} E_\alpha$$

is finite dimensional. We will now proceed as in the proof of Theorem 3.2. By Theorem 2.8, we have

$$(D^2\sigma, \sigma) - (\mathcal{R}\sigma, \sigma) - \|\nabla\sigma\|_X^2 = 0, \quad \forall \sigma \in \mathcal{D}(D).$$

Hence, for $\sigma \in H_\alpha$, $D\sigma = \lambda\sigma$, with $|\lambda| \leq \alpha$, so we get

$$\lambda^2 \|\sigma\|_X^2 - (\mathcal{R}\sigma, \sigma) - \|\nabla\sigma\|_X^2 = 0.$$

Since for some $k_1 \in \mathbb{R}$, $-\mathcal{R} \geq k_1 \mathrm{Id}$ on K, we have that, for any $\sigma \in H_\alpha$,

$$\lambda^2 \|\sigma\|_X^2 + \int_K k_1 \langle \sigma, \sigma \rangle \mathrm{d}v \geq \int_{X-K} \langle \mathcal{R}\sigma, \sigma \rangle \mathrm{d}v + \|\nabla\sigma\|_X^2.$$

As $k_0 \in \mathbb{R}$ is such that $k_0 \,\mathrm{Id} \leq \mathcal{R}$ on $X - K$, we get

$$\lambda^2 \|\sigma\|_X^2 + k_1 \|\sigma\|_K^2 \geq k_0 \|\sigma\|_{X-K}^2 + \|\nabla\sigma\|_X^2, \quad \forall \sigma \in H_\alpha.$$

Now replace $\|\sigma\|_{X-K}^2$ by $\|\sigma\|_X^2 - \|\sigma\|_K^2$ to get

$$\lambda^2 \|\sigma\|_X^2 + k_1 \|\sigma\|_K^2 \geq k_0 (\|\sigma\|_X^2 - \|\sigma\|_K^2) + \|\nabla\sigma\|_X^2, \quad \forall \sigma \in H_\alpha,$$
$$\lambda^2 \|\sigma\|_X^2 + (k_0 + k_1) \|\sigma\|_K^2 \geq k_0 \|\sigma\|_X^2 + \|\nabla\sigma\|_X^2, \quad \forall \sigma \in \sigma \in H_\alpha.$$

Because $\sigma \in H_\alpha$, $|\lambda| \leq \alpha$, and so $\lambda^2 \leq \alpha^2$, which implies

$$(k_0 + k_1) \|\sigma\|_K^2 \geq (k_0 - \alpha^2) \|\sigma\|_X^2 + \|\nabla\sigma\|_X^2, \quad \forall \sigma \in H_\alpha.$$

If we choose $\alpha > 0$ such that $\alpha^2 << k_0$, we have

$$\frac{k_0 - \alpha^2}{k_0 + k_1} \leq \frac{\|\sigma\|_K^2}{\|\sigma\|_X^2}, \quad \forall \sigma \in H_\alpha. \tag{3.2}$$

Now choose a parametrix Q for D, [21], so that $QD = \mathrm{Id} - T$, with T smoothing. Let ρ denote the restriction to K, and let $\breve{T} = \rho \circ T$. Then for any $\sigma \in H_\alpha$, we have

$$\breve{T}\sigma = \rho\sigma - \rho Q D\sigma.$$

Moreover, for any $\sigma \in H_\alpha$, $\|D\sigma\|_X \leq \alpha \|\sigma\|_X$. Set $q = \|Q\|$; from (3.2) we get

$$\|\breve{T}\sigma\|_X \geq \|\rho\sigma\|_X - \alpha q \|\sigma\|_X \geq (\alpha_1 - \alpha q) \|\sigma\|_X, \quad \forall \sigma \in H_\alpha,$$

where $\alpha_1 = \frac{k_0 - \alpha^2}{k_0 + k_1}$. Choosing α sufficiently small, $\|\breve{T}\sigma\|_X \geq \alpha_2 \|\sigma\|_X$, $\forall \sigma \in H_\alpha$. Therefore H_α is finite dimensional since $\tilde{T}$ is a compact operator. □

We will now define the analytical index of a pair of generalized Dirac operators that agree at infinity, thus generalizing a construction of Gromov and Lawson. We start with a non-compact complete orbifold which is sufficiently good at infinity. Let D_i and $D_i^{\pm}$ be the generalized Dirac operators on the orbifold X with coefficients in the Hermitian orbibundle E_i (with connection ∇^{E_i}), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood Ω of infinity. Assume that there exists a constant $k_0 > 0$ such that

$$\mathcal{R} \geq k_0 \mathrm{Id} \quad \text{on } \Omega.$$

Then we know from Theorem 3.2 that $\ker(D_i^{\pm}) < +\infty$. Thus the analytical index, of $D_i^{\pm}$, dimension of kernel minus dimension of cokernel, is finite, i.e., $\mathrm{ind}_a(D_i^{\pm}) < +\infty$, $i = 1, 2$.

Definition 3.6. Let X be a non-compact complete orbifold which is sufficiently good at infinity. Let D_i^+, E_i, $i = 1, 2$, be generalized Dirac operators, with coefficients in the Hermitian orbibundles E_i (with connection ∇^{E_i}), that agree on a neighborhood of infinity Ω. Then we define the analytical index, $\mathrm{ind}_a(D_1^+, D_2^+)$, of the pair (D_1^+, D_2^+) to be

$$\mathrm{ind}_a(D_1^+, D_2^+) = \mathrm{ind}_a(D_2^+) - \mathrm{ind}_a(D_1^+),$$

where $\mathrm{ind}_a(D_i^+)$, is the analytical index, dimension of kernel minus dimension of cokernel, of D_i^+, $i = 1, 2$; see Theorem 3.2.

4. The topological index

We will define here the topological index of a pair of generalized Dirac operators that agree at infinity, thus generalizing a construction of Gromov and Lawson, [15]. We start with a non-compact complete orbifold X which is sufficiently regular at infinity.

Let D_i and $D_i^{\pm}$ be the generalized Dirac operators on X with coefficients in the Hermitian orbibundle E_i (with connection ∇^{E_i}), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood of infinity.

We assume here that the orbifold X is sufficiently good at infinity. Recall that this means that, for any neighborhood of infinity Ω, we can chop off X along an orbifold hypersurface, O_Ω, which is the boundary of a neighborhood of infinity included in Ω. We also assume that the orbifold structure of X is of product type in a neighborhood of O_Ω. Note that sufficiently good at infinity implies sufficiently regular at infinity.

Chop off X along a hypersurface O, with $D_1 = D_2$ on the neighborhood of infinity of which O is boundary. Glue in an orbifold along O, so that the resulting space is a closed orbifold W. (This can be achieved, for example, by gluing another copy of the chopped off orbifold along O). We can assume that $E_1 = E_2$ on a neighborhood of O. Extend E_i to W, and call this extension $\breve{E}_i$, $i = 1, 2$. We can assume that on the glued in part, $\breve{E}_1 = \breve{E}_2$. Let $\breve{D}_i$ and $\breve{D}_i^{\pm}$ be the generalized Dirac operators on W with coefficients in $\breve{E}_i$, $i = 1, 2$. Define the topological index of the pair (D_1^+, D_2^+) by

Definition 4.1. Let X be a non-compact complete orbifold which is sufficiently good at infinity. Let E_i, D_i, D_i^+, $\breve{E}_i$, $\breve{D}_i^+$, $i = 1, 2$, be as above. Then we define the topological index, $\mathrm{ind}_t(D_1^+, D_2^+)$, of the pair (D_1^+, D_2^+) to be

$$\mathrm{ind}_t(D_1^+, D_2^+) = \mathrm{ind}_t(\breve{D}_2^+) - \mathrm{ind}_t(\breve{D}_1^+),$$

where $\mathrm{ind}_t(\breve{D}_i^+)$, $i = 1, 2$, is the topological index of Kawasaki of the operator $\breve{D}_i^+$ on the closed orbifold W, $i = 1, 2$ [21,22].

We will now give an explicit local description of the above index by using Kawasaki's formulas; see [21,22,9,31]. Let $\mathcal{U} = \{(\tilde{U}_i, G_i) | i \in I\}$, with $\tilde{U}_i / G_i = U_i$ and projection $\pi_i : \tilde{U}_i \to U_i$, be a standard orbifold atlas of W. Now $\tilde{U}_i^g$, the set of points fixed by g in $\tilde{U}_i$, for any $g \in G_i$, admits the action of the centralizer $Z_{G_i}(g)$ of g in G_i. If g and g' are conjugate in G_i, then $\tilde{U}_i^g$ and $\tilde{U}_i^{g'}$ are diffeomorphic via some element h in G_i, with $g' = hgh^{-1}$. So we can consider only one element for each conjugacy class in G_i. For each point $x \in W$, let $(1), \ldots, (h_x^{\rho_x})$ be all the conjugacy classes of the stabilizer G_x of x. Then we have a natural associated orbifold [22],

$$\hat{\Sigma} W = \left\{ (y, (h_y^j)) \mid y \in W, G_y \neq 1, j = 2, \ldots, \rho_y \right\}.$$

$\hat{\Sigma}W$ is stratified by orbit types; define

$$\hat{\hat{\Sigma}}W = W \cup \hat{\Sigma}W.$$

For short, we can rewrite $\hat{\hat{\Sigma}}W$ as

$$\hat{\hat{\Sigma}}W = \coprod_{j=1}^{q} \hat{\Sigma}^j W,$$

where $\hat{\Sigma}^1 W = W$, and so $\hat{\Sigma}^j W$ is the stratum corresponding to the j-th orbit type, $j = 1, \ldots, \ell$. In general, the action of $Z_{G_x}(h)$ on $\tilde{U}_x^h$ is not effective. The order of the subgroup of $Z_{G_x}(h)$ that acts trivially is called the multiplicity of $\hat{\Sigma}W$ at (x, h), and it is denoted by m $(x \in W$ and $h \in G_x)$. Hence to each connected component of $\hat{\Sigma}W$ we assign a certain constant multiplicity. The following theorem, the index theorem for generalized Dirac operators on closed orbifolds, was proved by Kawasaki in [20,21]; see also [9, Theorem 14.1], [5,31].

Theorem 4.2 *([21,9]). Let W be a closed orbifold. Let E be a Hermitian orbibundle (with connection ∇^E) on W and let D_E^+ be a generalized Dirac operator with coefficients in E. Then the (topological and analytical) index of D_E^+ on W is given in terms of local traces by*

$$\operatorname{ind}(D_E^+) = \int_{\hat{\hat{\Sigma}}W} \mathrm{d}\mu_{D_E^+}^{\Sigma} := \sum_{j=0}^{\ell} \frac{1}{m_j} \int_{\hat{\Sigma}^j W} \mathrm{d}\mu_{D_E^+}^{\Sigma, j},$$

where $\mathrm{d}\mu_{D_E^+}^{\Sigma,j}$ is the density on the j-th stratum of $\hat{\hat{\Sigma}}W$ associated (via parametrices) with the operator D_E^+, and m_j is the corresponding multiplicity function, $j = 1, \ldots, \ell$. (For the sake of simplicity, we assumed all the strata to be connected; in the general case there will be an additional summation over the connected components of the strata.)

Remark 4.3. For another formulation of Theorem 4.2 in terms of orbifold characteristic classes, see [22].

We will now use Theorem 4.2 to compute the topological index of a pair of generalized Dirac operators on a compact orbifold that is sufficiently good at infinity.

Theorem 4.4. *Let X, W, D_i, D_i^+, $\breve{D}_i^+$, E_i, $\breve{E}_i$, $i = 1, 2$, be as in Definition 4.1. Let Q_i be a semi-local parametrix for D_i^+, $i = 1, 2$, on W such that $Q_1 = Q_2$ in a neighborhood of infinity; write*

$$D_i^+ Q_i = \mathrm{Id} - T_i, \quad \textit{and} \quad Q_i D_i^+ = \mathrm{Id} - T_i', \quad i = 1, 2,$$

with T_i and T_i' the associated semi-local smoothing operators. Then the local trace functions of Kawasaki associated with T_1 and T_2 (and T_1' and T_2') coincide at infinity, and the topological index $\operatorname{ind}_t(D_1^+, D_2^+)$ of the pair (D_1^+, D_2^+) is given by,

$$\operatorname{ind}(\tilde{D}_E^+) = \int_{\hat{\hat{\Sigma}}X} \left(\mathrm{d}\mu_{T_2'}^{\Sigma} - \mathrm{d}\mu_{T_2}^{\Sigma}\right) - \int_{\hat{\hat{\Sigma}}X} \left(\mathrm{d}\mu_{T_1'}^{\Sigma} - \mathrm{d}\mu_{T_1}^{\Sigma}\right).$$

Proof. This theorem follows directly from Theorem 4.2. Indeed, if we denote by Ω the neighborhood of infinity where D_1 and D_2 coincide, we can cap off X along an orbifold O in Ω. Consider a parametrix Q_0 for the extension $\breve{D}_1 = \breve{D}_2$ on Ω. Now splice Q_0 onto Q_1 and Q_2, via a smooth function f which is 0 outside Ω, 1 on a neighborhood of infinity, and whose gradient is bounded by 1 in norm. For the rest of the proof we can proceed exactly as in [15, Proof of Proposition 4.6]. Note that we need to use Kawasaki's local trace formulas. □

Remark 4.5. The topological index of the pair, $\operatorname{ind}_t(D_1^+, D_2^+)$, is independent of the extension.

5. The analytical index: Computations

We will now detail some local properties of the trace of a generalized Dirac operator D, thus extending to orbifolds some results of Gromov and Lawson; see [15]. Assume that there exists a constant $k_0 > 0$ such that

$$\mathcal{R} \geq k_0 \mathrm{Id} \quad \text{on } \Omega,$$

with $\Omega = X - K$ as before. Because of Theorem 2.8, our assumption implies

$$\|D\sigma\|_X \geq c\|\sigma\|_X, \quad \forall \sigma \in \mathcal{C}_c^\infty(X, \mathcal{S}) \quad \text{such that } \operatorname{supp}(\sigma) \cap K = \emptyset. \tag{5.1}$$

Our goal in this section is to explicitly compute the index trace of D in terms of local data, by using techniques of Gromov and Lawson and of Anghel; see [15,3]. This will enable us to prove, in Section 6, the relative index theorem by localization. Also, for the sake of simplicity, in this section we will take $E = \mathbb{C}$. So D will be defined on sections of the Spin^c bundle $\mathcal{S}$. Start with a parametrix Q_0 of D on X. Then

$$DQ_0 = \mathrm{Id} - R, \quad \text{and} \quad Q_0 D = \mathrm{Id} - R', \quad \text{on } X,$$

where R and R' are not necessarily trace class (as instead happens in the closed orbifold case). We will now replace Q_0, outside a compact set K_0, with a Green operator associated with a suitable extension of D. Specifically, set $\Omega_0 = X - K_0$, with Ω_0 a domain with smooth boundary included in Ω. Let D_{Ω_0} be the graph closure of the restriction of D to $\mathcal{C}_c^\infty(\Omega_0, \mathcal{S})$ in $\mathcal{L}^2(X, \mathcal{S})$. Let P_{Ω_0} denote the orthogonal projection from $\mathcal{L}^2(X, \mathcal{S})$ to

$$H_{\Omega_0} = \{\sigma \in \mathcal{L}^2(\Omega_0, \mathcal{S}) | D|_{\Omega_0}(\sigma|_{\Omega_0}) = 0 \text{ distributionally}\}.$$

Note that, by (5.1), H_{Ω_0} is a closed subspace of $\mathcal{L}^2(X, \mathcal{S})$.

Theorem 5.1. *Let X be a non-compact complete orbifold which is sufficiently good at infinity. Let D, $\mathcal{S}$, D_{Ω_0}, P_{Ω_0}, be as above. Then the following two equations:*

$$D_{\Omega_0} G_{\Omega_0} = \mathrm{Id} - P_{\Omega_0} \quad \text{on } \mathcal{L}^2(X, \mathcal{S}) \tag{5.2}$$

$$G_{\Omega_0} D_{\Omega_0} = \mathrm{Id} \quad \text{on the minimal domain of } D_{\Omega_0} \tag{5.3}$$

define a bounded operator G_{Ω_0}: $\mathcal{L}^2(X, \mathcal{S}) \to \mathcal{L}^2(X, \mathcal{S})$, the Green operator of D_{Ω_0}.

Proof. Eq. (5.1) shows that D_{Ω_0} is 1:1 and has closed range, call it $\operatorname{ran}(D_{\Omega_0})$. Hence $\mathcal{L}^2(X, \mathcal{S})$ decomposes orthogonally as

$$\mathcal{L}^2(X, \mathcal{S}) = H_{\Omega_0} \oplus \operatorname{ran}(D_{\Omega_0}).$$

Thus G_{Ω_0} can be defined to be zero on H_{Ω_0} and $D_{\Omega_0}^{-1}$ on $\operatorname{ran}(D_{\Omega_0})$. G_{Ω_0} is bounded because of (5.1). □

Since H_{Ω_0} is a closed subspace of $\mathcal{L}^2(X, \mathcal{S})$, we can define the orthogonal projection on H_{Ω_0} as the Bergman kernel operator

$$\mathcal{P}_H(x, y) = \sum_m \sigma_m(x) \otimes \sigma_m^*(y), \quad \forall x, y, \in \Omega_0,$$

where $\{\sigma_m\}$, $m \in \mathbb{N}$, is an orthonormal basis of H_{Ω_0}.

Theorem 5.2. *Let X be a non-compact complete orbifold which is sufficiently good at infinity. Let D, $\mathcal{S}$, D_{Ω_0}, P_{Ω_0}, be as above. Then the following Bergman kernel:*

$$\mathcal{P}_H(x, y) = \sum_m \sigma_m(x) \otimes \sigma_m^*(y), \quad \forall x, y, \in \Omega_0$$

of the projection operator P_{Ω_0} converges (on lifts of local charts; see the proof for details) uniformly in the norm $\mathcal{C}^k$ on compact subsets, $k \in \mathbb{N}$.

Proof. (In this proof ˜ means lifted to the local orbifold charts.) Because X is sufficiently good at infinity, we can assume that there exists a smooth exhaustion function F on X, such that $\Omega_0 = F(t_0) = \{x \in X | F(x) > t_0\}$. To show the convergence of the Bergman kernel $\mathcal{P}_H$ we proceed in the following way. First of all, set $\mathcal{B}_N = \sum_{m=1}^N \sigma_m(x) \otimes \sigma_m^*(y)$. Obviously B_N is a finite rank operator in $\mathcal{L}^2(X, \mathcal{S})$. Now, for any $\phi \in \mathcal{L}^2(X, \mathcal{S})$, $B_N(\phi) \to B(\phi)$. In particular $B_N(\phi) \to B(\phi)$ in $\mathcal{D}'$ (as distributions). Using local charts, $B_N \to B$ weakly in $\mathcal{L}_{G_U}(\mathcal{D}_{\tilde{y}}, \mathcal{D}'_{\tilde{x}})$ for any two points $\tilde{x}$, $\tilde{y}$ of an orbifold chart $(\tilde{U}, G_U)$ projecting to x and y respectively. (By using a partition of unity, the convergence of distributions is a local property; use a standard orbifold atlas.) Also, for the sake of simplicity, we have taken x and

y to be in the same orbifold chart. Now, P_{Ω_0}, which in this proof we will call P, is a bounded operator on $\mathcal{L}^2(X, \mathcal{S})$, and so is also continuous as an operator $\mathcal{D} \to \mathcal{D}'$. Hence P has a G_U-invariant distributional kernel $p(\tilde{x}, \tilde{y})$ in the sense of Schwartz; cf. [4,25,26]. By the Schwartz kernel theorem, $B_N \to \mathcal{P}_H$ locally as a G_U-invariant distribution on $\tilde{U} \times \tilde{U}$. (This follows as in the manifold case; see [4, pg. 51].) So

$$\mathcal{P}_H(\tilde{x}, \tilde{y}) = \sum_{m=1}^{+\infty} \sigma_m(\tilde{x}) \otimes \sigma_m^*(\tilde{y})$$

G_U-invariantly, on local charts. Now note that $\mathcal{P}_H$ and B_N satisfy the elliptic equation $\tilde{D}^2 = 0$. Therefore by applying Theorem 3.1 locally, we have the required uniform convergence. □

The kernel $\mathcal{P}_H$ has the strong finiteness property proved below. Let $F : X \to \mathbb{R}^+$ be a smooth exhaustion function as above. In particular we assume that $\Omega_0 = F^{-1}(t_0, +\infty)$, $K = F^{-1}[0, t_0]$. Let $X(t) = \{x \in X | F(x) > t\}$; then $\Omega_0 = X(t_0)$. We have, cf. [15, Lemma 4.20] for the manifold case,

Theorem 5.3. *Let X be a non-compact complete orbifold which is sufficiently good at infinity. Let D, $\mathcal{S}$, D_{Ω_0}, P_{Ω_0}, $\mathcal{P}_H(x, x)$ be as above. Then for any $t > t_0$,*

$$\int_{X(t)} \mathcal{P}_H(x, x) < +\infty.$$

Proof. For fixed $t > t_0$ (t near t_0), choose s so that $t_0 < s < t$, and consider the compact "annulus" $A =$ closure $[X(s) - X(t)]$. Since $\mathcal{P}_H(x, x)$ converges uniformly in the $\mathcal{C}^k$, $k \in \mathbb{N}$, norm, on compact subsets, we can assume that it converges in the Sobolev norm on A, that is,

$$\sum_m \|\sigma_m\|_{1,A} < +\infty. \tag{5.4}$$

We now claim that there exists a constant c so that, for each $\sigma \in H_{\Omega_0}$,

$$\|\sigma_m\|^2_{1,X(t)} \leq c\|\sigma\|^2_{1,A}. \tag{5.5}$$

To do so, choose a cut-off function $f \in \mathcal{C}^\infty(\Omega_0)$ such that $0 \leq f \leq 1$, $f = 1$ on $X(t)$, and $f = 0$ on $\Omega_0 - X(s)$. Clearly there exists a $c_0 \in \mathbb{R}^+$ such that $\|\nabla f\|_X < c_0$. Applying the local identity

$$\nabla(f\sigma) = \nabla(f)\sigma + f\nabla\sigma$$

and $D^2 = \nabla^*\nabla + \mathcal{R}$, we have, for every $\sigma \in H_{\Omega_0}$,

$$\begin{aligned}
0 &= (D^2_{\Omega_0}\sigma, f^2\sigma) = ((\nabla^*\nabla + \mathcal{R})\sigma, f^2\sigma)\\
0 &= (\nabla^*\nabla\sigma, f^2\sigma) + (\mathcal{R}\sigma, f^2\sigma)\\
0 &= (\nabla\sigma, \nabla(f^2\sigma)) + (\mathcal{R}\sigma, f^2\sigma)\\
0 &= (\nabla\sigma, 2f(\nabla f)\sigma) + (\nabla\sigma, f^2(\nabla\sigma)) + (\mathcal{R}\sigma, f^2\sigma)\\
0 &= 2(f\nabla\sigma, (\nabla f)\sigma) + (f\nabla\sigma, f(\nabla\sigma)) + (\mathcal{R}\sigma, f^2\sigma).
\end{aligned}$$

Since $\langle \mathcal{R}\alpha, \alpha\rangle \geq k_0\langle\alpha, \alpha\rangle$ in Ω_0, we have

$$\|\nabla\sigma\|^2_{X(t)} + k_0\|\sigma\|^2_{X(t)} \leq 2|(f\nabla\sigma, (\nabla f)\sigma)| \leq 2c_0\|\nabla\sigma\|_A\|\sigma\|_A \leq c_0(\|\nabla\sigma\|^2_A + \|\sigma\|^2_A).$$

Then

$$\|\sigma\|^2_{1,X(t)} \leq c_0\left(1 + \frac{1}{k_0}\right)\|\sigma\|^2_{1,A}.$$

We thus proved our claim (5.5). Now the lemma is proved by combining (5.4) and (5.5). □

6. The relative index theorem

We will prove here that the analytical index of a pair of generalized Dirac operators that agree at infinity is equal to its analytical index, thus generalizing to orbifolds the main theorem of Gromov and Lawson given in [15]. Let X be a non-compact complete orbifold which is sufficiently good at infinity. Assume that there exists a constant $k_0 > 0$ such that

$$\mathcal{R} \geq k_0 \,\mathrm{Id} \quad \text{on } \Omega, \qquad K = X - \Omega,$$

with Ω a domain with smooth boundary, and $\mathcal{R}$ as in Proposition 2.7. Let D_i and $D_i^{\pm}$ be the generalized Dirac operators on X with coefficients in the Hermitian orbibundle E_i (with connection ∇_i^E), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood Ω of infinity. Let G_i, $i = 1, 2$, be the Green operator associated with D_i^+, $i = 1, 2$; see Theorem 3.5. In particular, D_i^+ and G_i satisfy the following equations:

$$D_i^+ G_i = \mathrm{Id} - P_i^-, \quad \text{and} \quad G_i D_i^+ = \mathrm{Id} - P_i^+, \quad i = 1, 2,$$

where $P_i^{\pm} : \mathcal{L}^2(X, \mathcal{S}) \to \mathcal{L}^2(X, \mathcal{S})$ is the projection onto the finite dimensional space $\ker D_i^{\pm}$. On Ω, $D_1 = D_2$. Restrict G_i to Ω by defining $\hat{G}_i = \chi G_i \chi$, $i = 1, 2$, where χ is the characteristic function of Ω. The difference $\hat{G}_2 - \hat{G}_1$ satisfies

$$D^+(\hat{G}_2 - \hat{G}_1) = \hat{P}_1^- - \hat{P}_2^-,$$

where $D_1 = D_2 = D$ on Ω, and $\hat{P}_i^{\pm} = \chi P_i^{\pm} \chi$, $i = 1, 2$, is a finite range operator. This implies, as in [15], that the range of $(\hat{G}_2^+ - \hat{G}_1^+)$ is nearly contained in the kernel H_Ω of D^+ on Ω, with the notation as in Proposition 2.7. In other words, let V be $\ker(\hat{P}_2^+ - \hat{P}_1^+)$. Then V is a subspace of finite codimension in $\mathcal{L}^2(X, \mathcal{S}^+)$, and

$$(\hat{G}_2 - \hat{G}_1)V \subseteq \ker(D^+).$$

Theorem 6.1 (*Cf. [15, Lemma 4.28] for the Manifold Case.*). *Let X, D_i, G_i, $\hat{G}_i$, $i = 1, 2$, be as above. Then the local trace density function of $(\hat{G}_2 - \hat{G}_1)$ is integrable at infinity.*

Proof. Recall that $\Omega = X - K$, with K compact. Choose $\Omega' = X - L$, with L compact such that $K \subseteq \mathrm{int}(L)$. By Theorem 5.3, if P_Ω denotes the orthogonal projection onto the kernel H_Ω of D^+ on Ω, and $\mathcal{P}_\Omega$ is its associated trace density, then

$$\int_{\Omega'} \mathcal{P}_\Omega(x) \mathrm{d}v < +\infty.$$

Set $Z = (\hat{G}_2 - \hat{G}_1)$. Then

$$\mathrm{range}(Z) \subseteq H_\Omega + F,$$

where F is a finite dimensional subspace of $\mathcal{L}^2(X, \mathcal{S}^-)$, by the discussion preceding the statement of Theorem 6.1. Let $\{\sigma_m\}$, $m \in \mathbb{N}$, be an orthonormal basis of $H_\Omega + F$, such that $\{\sigma_m\}$, $m = N, N+1, \ldots$, is an orthonormal basis of H_Ω. Then the Schwartzian kernel of Z can be written as

$$K^Z(x, y) = \sum_m \sigma_m(x) \otimes (Z^* \sigma_m)(y), \quad \forall x, y, \in \Omega,$$

where Z^* denotes the adjoint of Z. The local trace function $\mathcal{P}_Z$ of Z satisfies

$$|\mathcal{P}_Z(x)| \leq \sum_m |\langle \sigma_m(x), (Z^* \sigma_m)(x) \rangle|, \quad \forall x \in \Omega.$$

Let $Z' = \chi' Z \chi'$, with χ' the characteristic function of Ω', denote the restriction of Z to Ω'; note that $\|Z'\| \leq \|Z\|$. Then

$$\begin{aligned}\int_{\Omega'} |\mathcal{P}_Z(x)| &\le \sum_m \int_{\Omega'} |\langle Z\sigma_m(x), \sigma_m(x)\rangle| \mathrm{d}x \\ &\le \sum_m \|Z\sigma_m\|_{\Omega'} \|\sigma_m\|_{\Omega'} \\ &\le \sum_m \|Z'\sigma_m\|_{\Omega'} \|\sigma_m\|_{\Omega'} \\ &\le \|Z\| \sum_m \|(\sigma_m)\|^2_{\Omega'} \\ &\le \|Z\| \left(\sum_m^{N-1} \|(\sigma_m)\|^2_{\Omega'} + \int_{\Omega'} |\mathcal{P}_Z| \right) < +\infty. \quad \square\end{aligned}$$

We are now in a position to state and prove our relative index theorem.

Theorem 6.2. *Let X be a non-compact complete orbifold which is sufficiently good at infinity. Let D_i, $D_i^{\pm}$ be the generalized Dirac operators on X with coefficients in the Hermitian orbibundle E_i (with connection ∇_i^E), $i = 1, 2$. Suppose that $D_1 = D_2$ in a neighborhood Ω of infinity with smooth boundary. Assume that there exists a constant $k_0 > 0$ such that*

$$\mathcal{R} \ge k_0 \,\mathrm{Id} \quad \text{on } \Omega$$

where $\mathcal{R}$ is given as in Section 5. Then the topological index (as in Definition 4.1) and the analytical index (as in Definition 3.6) of the pair (D_1^+, D_2^+) coincide, that is

$$\mathrm{ind}_t(D_1^+, D_2^+) = \mathrm{ind}_a(D_1^+, D_2^+).$$

The proof of Theorem 6.2 will occupy the rest of this section.

Proof (*Cf. [15, Proof of Theorem 4.18]*). We will now construct parametrices Q_j for D_j^+, $j = 1, 2$, by cutting off the Green operators G_i, $i = 1, 2$, of Theorem 3.5 in a neighborhood of the diagonal. (Notation as at the beginning of this section.) Choose $\psi \in \mathcal{C}^\infty(X \times X)$ with support in a small neighborhood of the diagonal, so that $0 \le \psi \le 1$ and $\psi = 1$ near the diagonal. Note that ψ can in particular be lifted, one variable at the time, to a G_U-invariant function in $\tilde{\psi}$ in $\mathcal{C}^\infty(\tilde{U} \times \tilde{U})$, for any local orbifold chart $(\tilde{U}, G_U)$. Let R_i, $i = 1, 2$, be the operator whose Schwartzian kernel is locally given by

$$K^i(\tilde{x}, \tilde{y}) = \tilde{\psi} K^{G_i}(\tilde{x}, \tilde{y}).$$

The operator R_i, $i = 1, 2$, is a semi-local parametrix for D_i^+ with

$$D_i^+ R_i = \mathrm{Id} - S_i^-, \qquad R_i D_i^+ = \mathrm{Id} - S_i^+, \quad i = 1, 2.$$

As $R_i = G_i$ near the diagonal, the local traces associated with $S_i^{\pm}$, $i = 1, 2$, satisfy

$$t^{S_i^+} = t^{i^+}, \qquad t^{S_i^-} = t^{i^-},$$

where $t^{i^{\pm}}$ is the trace associated with $P_i : \mathcal{L}^2(X, \mathcal{S}) \to \mathcal{L}^2(X, \mathcal{S})$, the projection onto the finite dimensional space $\ker D_i^+$ and $\ker D_i^- = \operatorname{coker} D_i^+$, $i = 1, 2$. From Theorem 4.4, to compute the topological index we need a pair of semi-local parametrices that agree in a neighborhood of infinity. We will do this by splicing R_2 onto R_1 in Ω as follows. Let $\{b_k\}$, $k \in \mathbb{N}$, be a sequence of functions as in Proposition 2.4. We assume that, for k sufficiently large, $b_k = 1$ on K. Moreover, since $\mathrm{supp}(b_k) \subseteq \overline{B}_{2k}$, we can assume that $b_k = 0$ in a neighborhood of infinity. We claim that each b_k can be approximated by a smooth function f_k such that $f_k = 1$ on K, $\mathrm{supp}(f_k) \subseteq \overline{B}_{2k}$. This claim is proven in the following way. $\mathrm{Supp}(b_k)$ can be covered by the union of finitely many orbifold charts, say U_j with $\tilde{U}_j / G_j = U_j$, $j = 1, \dots, \ell$. If $\tilde{b}_{k,s}$ is the lift of a smooth approximation of b_k on U_s, obtained G_s-invariantly, then $\tilde{f}_k = \sum_s \tilde{\eta}_s \tilde{b}_{k,s}$ is a smooth approximation of b_k, $k \in \mathbb{N}$. Now define a sequence of semi-local parametrices $R_{2,k}$ for D_2^+ by

$$R_{2,k} = f_k R_2 + (1 - f_k) R_1$$

(recall that on Ω, $D_1^+ = D_2^+$). Moreover,

$$D_2^+ R_{2,k} = \mathrm{Id} - S_{2,k}^-, \qquad R_{2,k} D_2^+ = \mathrm{Id} - S_{2,k}^+,$$

with

$$\begin{aligned} S_{2,k}^- &= f_k S_2^- + (1 - f_k) S_1^- + \nabla(f_k)(R_2 - R_1) \\ S_{2,k}^+ &= f_k S_2^+ + (1 - f_k) S_1^+ \end{aligned} \tag{6.1}$$

where $S_i^\pm$, $i = 1, 2$, satisfy

$$\begin{aligned} D_1^+ R_1 &= \mathrm{Id} - S_1^-, \qquad R_1 D_1^+ = \mathrm{Id} - S_1^+, \\ D_2^+ R_2 &= \mathrm{Id} - S_2^-, \qquad R_2 D_2^+ = \mathrm{Id} - S_2^+. \end{aligned}$$

Now by Theorem 4.4 applied to the pair of parametrices $R_{2,k}$, R_1, we get

$$\mathrm{ind}_t(D_1^+, D_2^+) = \int_{\hat{\hat{\Sigma}} X} \left(\mathrm{d}\mu^{S_{2,k}^+} - \mathrm{d}\mu^{S_{2,k}^-}\right) - \int_{\hat{\hat{\Sigma}} X} \left(\mathrm{d}\mu^{S_1^+} - \mathrm{d}\mu^{S_1^-}\right).$$

By (6.1) this latter expression is equal to

$$\begin{aligned} &= \int_{\hat{\hat{\Sigma}} X} f_k \left(\mathrm{d}\mu^{S_2^+} - \mathrm{d}\mu^{S_2^-} - \mathrm{d}\mu^{S_1^+} + \mathrm{d}\mu^{S_1^-}\right) - \int_{\hat{\hat{\Sigma}} \Omega} \mathrm{d}\mu^{\nabla f_k (R_2 . R_1)} \\ &= \int_{\hat{\hat{\Sigma}} X} f_k \left(\mathrm{d}\mu^{P_2^+} - \mathrm{d}\mu^{P_2^-} - \mathrm{d}\mu^{P_1^+} + \mathrm{d}\mu^{P_1^-}\right) - \int_{\hat{\hat{\Sigma}} \Omega} \mathrm{d}\mu^{\nabla f_k (R_2 . R_1)}, \end{aligned}$$

where $P_i^\pm$ is the projection onto the kernel of $D_i^\pm$, $i = 1, 2$. Now the Schwartz kernel of $Z_k = \nabla f_k (R_2 - R_1)$ on a local chart is

$$K^{Z_k}(\tilde{x}, \tilde{y}) = \nabla \tilde{f}_k K^{R_2 - R_1}(\tilde{x}, \tilde{y}).$$

Near the diagonal, $R_i = G_i$, $i = 1, 2$, and so

$$|\mathrm{d}\mu^{Z_k}(x)| \le \|\nabla f_k\| |\mathrm{d}\mu^{Z}(x)|,$$

where $Z = \hat{G}_2 - \hat{G}_1$ as above. By integrating on $\hat{\hat{\Sigma}} \Omega$, since $\mathrm{supp}(\nabla f_k) \subseteq \Omega$, we have

$$\int_{\hat{\hat{\Sigma}} \Omega} \mathrm{d}\mu^{\nabla f_k (R_2 - R_1)} = \int_{\hat{\hat{\Sigma}} \Omega'} \mathrm{d}\mu^{\nabla f_k (R_2 - R_1)} \le \int_{\hat{\hat{\Sigma}} \Omega'} \|\nabla f_k\| |K^{R_2 - R_1}| = \int_{\hat{\hat{\Sigma}} \Omega'} \|\nabla f_k\| |K^{Z}|.$$

Since $\int_{\hat{\hat{\Sigma}} \Omega'} |K^Z| < +\infty$ by Lemma 5.3, our result follows if we show that

$$\sup_{\Omega} \|\nabla f_k\| \to 0 \quad \text{as } k \to +\infty.$$

To show this latter claim, we will show that $\sup_{\Omega} \|\nabla f_k\| \le \frac{M}{k}$ for $k \in \mathbb{N}$. For, the compact subset $\mathrm{supp}(b_k)$ can be covered by finitely many orbifold charts. On each of these charts, we can assume that we have an approximation with $\sup \|\nabla f_k\| \le \frac{M}{k}$, $k \in \mathbb{N}$, because we first approximate on its lift, and then average over the chart group. Each function (and each function's gradient) on average will have the same sup. Moreover, on the lift of each chart we can assume that the gradient of the distance function is bounded by $\frac{M}{k}$ except on a (singular) set of zero measure and codimension at least 2. Hence by possibly performing a cutting and limiting procedure around the singular locus, a smooth approximation with the required bound can be found. □

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